

On the angular momentum of photons: canonical quantization of radiation field at the inner reference frame

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Abstract

The properties of the angular momentum of photons are investigated in what is called the Jones representation. Different from the Maxwell wavefunction that is defined in the laboratory reference frame and is constrained by the transversality condition, the wavefunction in the Jones representation, the Jones wavefunction, is defined in the so-called inner reference frame and does not suffer from any constraints. The position with respect to the inner reference frame is canonically conjugate to the momentum. Such a characteristic allows us to quantize the Jones wavefunction in a canonical way. The origin of the inner reference frame with respect to the laboratory reference frame is represented by an operator that is dependent on the intrinsic degree of freedom, the helicity. A new degree of freedom that appears as a unit vector is also identified to specify the operator. For a radiation field that is described by a particular Maxwell wavefunction, the new degree of freedom is analogous to the classical gauge degree of freedom; the Jones wavefunction is analogous to the gauge potentials. For an eigen excitation of the radiation field, on the other hand, the new degree of freedom plays the role of specifying its barycenter with respect to the laboratory reference frame.

After expressed in the Jones representation, the spin and the orbital angular momentum about the origin of the laboratory reference frame are shown to obey the commutation relations that were obtained by van Enk and Nienhuis [J. Mod. Opt. **41**, 963 (1994)] from a consideration of the second quantization.

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I. INTRODUCTION

This paper is concerned with an unusual quantum degree of freedom that arises from the constraint of transversality condition and its implication to the angular momentum of the photon. It is well known [1–6] that the properties of the angular momentum of the photon, especially the separability of the spin from the orbital angular momentum (OAM), have been a long-standing controversial topic. Since the seminal work of Allen and his collaborators [7], there appeared more and more evidences that demonstrate distinctive differences between the spin and the OAM. It was found [6, 8] that the spin and the OAM can induce different effects in the interaction with tiny birefringent particles trapped off axis in optical tweezers. The spin makes the particle rotate about its own axis and the OAM makes the particle rotate about the axis of the optical beam. The conversion from the spin to the OAM was also observed in anisotropic [9], isotropic [10], and nonlinear [11] media. At the same time, a lot of theoretical effort was made in an attempt to separate the spin from the OAM. In their creative paper [7], Allen et. al. showed with paraxial Laguerre-Gaussian beams that the OAM is carried by a spiral wavefront and the spin is carried by the helicity. However, taking that result as a criterion, the separation between the spin and the OAM was shown [12, 13] to be impossible beyond the paraxial approximation. Very recently [14], on the basis of quite a general criterion that the spin is independent of the choice of reference point and the OAM is dependent on the choice of reference point, the total angular momentum of an arbitrary radiation field was rigorously separated into the spin and orbital parts.

Nonetheless, up to now there has not been a well accepted quantum theory to explain the separation of the spin from the OAM. As is well known in the first-quantization theory [1], the operator of spin and the operator of OAM about the origin of a laboratory reference frame can be written as

$$\hat{\mathbf{S}} = \hbar \hat{\boldsymbol{\Sigma}}, \quad (1a)$$

$$\hat{\mathbf{L}} = -\hbar \mathbf{k} \times \hat{\mathbf{X}}, \quad (1b)$$

respectively, where $(\hat{\boldsymbol{\Sigma}}_k)_{ij} = -i\epsilon_{ijk}$ with ϵ_{ijk} the Levi-Civita pseudotensor, \mathbf{k} is the wavevector,

$$\hat{\mathbf{X}} = i\nabla \quad (2)$$

is the operator of the position with respect to the laboratory reference frame, and ∇ is

the gradient operator in the wavevector space (\mathbf{k} -space). They act on the \mathbf{k} -space vector wavefunction $\mathbf{f}(\mathbf{k}, t)$ which satisfies the Schrödinger equation

$$i\frac{\partial \mathbf{f}}{\partial t} = \omega \mathbf{f} \quad (3)$$

and is constrained by the transversality condition

$$\mathbf{f}^\dagger \mathbf{w} = 0, \quad (4)$$

where $\omega = ck$ is the angular frequency, $k = |\mathbf{k}|$ is the wave number, $\mathbf{w} = \frac{\mathbf{k}}{k}$ is the unit wavevector, the convention of matrix multiplication is used for the inner product of two vectors, and the superscript \dagger denotes the conjugate transpose. The angular frequency ω plays the role of the Hamiltonian. Equation (3) together with the transversality condition (4) is strictly equivalent to the system of free-space Maxwell's equations [1, 3]. The spin and OAM operators were believed [4, 15] to obey the following standard commutation relations of the angular momentum,

$$[\hat{S}_i, \hat{S}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{S}_k, \quad (5a)$$

$$[\hat{L}_i, \hat{L}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{L}_k, \quad (5b)$$

respectively, and therefore to be the generators of certain kinds of spatial rotation [2, 3]. Observing [3, 4] that the rotation generated by either of them might ruin the transversality of the wavefunction, the separation of the spin from the OAM was thought to be physically meaningless. In 1994, van Enk and Nienhuis [4] made a valuable attempt to distinguish the spin and the OAM in a framework of the second quantization. They showed that their second-quantization operators \mathbf{S}^{en} and \mathbf{L}^{en} for the spin and the OAM, respectively, do not obey the standard commutation relations. The commutation relations that they found are as follows,

$$[S_i^{en}, S_j^{en}] = 0, \quad (6a)$$

$$[L_i^{en}, L_j^{en}] = i\hbar \sum_k \epsilon_{ijk} (L_k^{en} - S_k^{en}). \quad (6b)$$

What is worth noting is that they made every effort to vindicate the commutation relations (5) for the operators $\hat{\mathbf{S}}$ and $\hat{\mathbf{L}}$. This is unsatisfactory, because their operators \mathbf{S}^{en} and \mathbf{L}^{en}

are just the second-quantization counterparts of operators $\hat{\mathbf{S}}$ and $\hat{\mathbf{L}}$, respectively. After all, commutation relations in quantum theory mean nothing but quantization conditions [16].

The purpose of this paper is to advance a quantum theory for the angular momentum of the photon. The main idea is to explore the role that the constraint of transversality condition plays from quite a different point of view. A quantum representation of radiation field, called the Jones representation, is introduced from the transversality condition. Different from the vector wavefunction \mathbf{f} that is defined in the laboratory reference frame, the wavefunction in the Jones representation, called the Jones wavefunction, is defined in the so-called inner reference frame. The benefit to introduce the Jones representation is that the Jones wavefunction can be canonically quantized. But, unexpectedly, the inner reference frame is not related to the laboratory reference frame simply by a spatial translation. Its origin with respect to the laboratory reference frame is represented by an operator that is dependent on the intrinsic degree of freedom, the helicity. What is more unexpected is that the transversality condition alone is not able to fully determine the operator. In order to do so, a complementary degree of freedom that appears as a unit vector \mathbf{I} is identified. For a radiation field that is described by a particular vector wavefunction \mathbf{f} , the new degree of freedom to specify the inner reference frame is analogous to the classical gauge degree of freedom; the Jones wavefunction in the inner reference frame is analogous to the classical gauge potentials. On the other hand, for an eigen excitation of the radiation field, the new degree of freedom plays the role of specifying its barycenter or center of mass with respect to the laboratory reference frame. This is because the inner reference frame in this case reduces to its barycenter reference frame in the sense that its barycenter is located at the origin of the inner reference frame.

It is found in the Jones representation that the spin lies entirely along the wavevector direction and that the OAM about the origin of the laboratory reference frame depends not only on the new degree of freedom \mathbf{I} but also on the helicity. This helps us to understand why the total angular momentum can not be generally separated [12] into helicity-dependent spin part and helicity-independent orbital part. It is shown in the Jones representation that the spin and the OAM do not obey the standard commutation relations (5) and that the commutation relations (6) found by van Enk and Nienhuis do not necessarily rely on the second quantization.

This paper is arranged as follows. It is shown in Section II that the transversality condi-

tion (4) leads to a quasi unitary matrix (10), the conjugate transpose of which transforms the vector wavefunction into the Jones wavefunction via Eq. (13). Different from the vector wavefunction, the Jones wavefunction is free of any constraints such as Eq. (4). It is found in Section III that the Jones wavefunction is defined in the inner reference frame. The origin of the inner reference frame with respect to the laboratory reference frame is represented by an operator that is expressible in terms of the quasi unitary matrix via Eq. (16). The position with respect to the inner reference frame is canonically conjugate to the momentum. In Section IV the Jones wavefunction is quantized using the canonical variables as well as the helicity. Three kinds of complete sets of mode functions are given. In Section V the complementary degree of freedom \mathbf{I} to specify the inner reference frame is identified. At last the properties of photon's angular momenta, especially their commutation relations, are discussed in Section VI. Section VII concludes the paper with remarks.

II. FROM TRANSVERSALITY TO QUASI UNITARY TRANSFORMATION

The transversality condition (4) means that the vector wavefunction \mathbf{f} can be expressed in terms of two orthonormal base vectors as

$$\mathbf{f} = \mathbf{u}f_1 + \mathbf{v}f_2, \quad (7)$$

where for simplicity the base vectors \mathbf{u} and \mathbf{v} are supposed to be real-valued, forming a righthand Cartesian coordinate system with \mathbf{w} ,

$$\mathbf{v}^T \mathbf{u} = \mathbf{w}^T \mathbf{v} = \mathbf{u}^T \mathbf{w} = 0, \quad (8a)$$

$$\mathbf{u}^T \mathbf{u} = \mathbf{v}^T \mathbf{v} = 1, \quad (8b)$$

$$\mathbf{u} \times \mathbf{v} = \mathbf{w}, \quad (8c)$$

and the superscript T denotes the transpose. For the purpose of this paper, I convert Eq. (7) into a compact form,

$$\mathbf{f} = \varpi(\mathbf{w})\tilde{f}, \quad (9)$$

where

$$\varpi(\mathbf{w}) = (\mathbf{u} \ \mathbf{v}) \quad (10)$$

is a 3-by-2 matrix consisting of two column vectors \mathbf{u} and \mathbf{v} and $\tilde{f} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ is a two-component wavefunction. It is noted that the matrix ϖ is dependent on the unit wavevector \mathbf{w} . Substituting Eq. (9) into Eq. (3), we get

$$i\frac{\partial \tilde{f}}{\partial t} = \omega \tilde{f}, \quad (11)$$

which is the Schrödinger equation for the two-component wavefunction \tilde{f} . Eq. (11) together with Eq. (9) is equivalent to the system of free-space Maxwell's equations.

The matrix ϖ in Eq. (9) performs a quasi unitary transformation in the following sense. Firstly, ϖ transforms a two-component wavefunction into a vector wavefunction. It satisfies

$$\varpi^\dagger \varpi = I_2 \quad (12)$$

so that keeps unchanged the norm of a wavefunction under the transformation,

$$\mathbf{f}^\dagger \mathbf{f} = \tilde{f}^\dagger \tilde{f},$$

where I_2 denotes the 2-by-2 unit matrix. Secondly, from Eqs. (9) and (12) is obtained

$$\tilde{f} = \varpi^\dagger \mathbf{f}, \quad (13)$$

showing that the matrix ϖ^\dagger transforms a vector wavefunction into a two-component wavefunction. Substituting it into Eq. (9), we find

$$\varpi \varpi^\dagger \mathbf{f} = \mathbf{f}.$$

Noticing that $\varpi \varpi^\dagger$ acts on the vector wavefunction, we may rewrite the above equation simply as

$$\varpi \varpi^\dagger = I_3, \quad (14)$$

due to the arbitrariness of the vector wavefunction \mathbf{f} , where I_3 denotes the 3-by-3 unit matrix. Eqs. (12) and (14) express the quasi unitarity [17] of the transformation matrix ϖ . ϖ^\dagger is the Moore-Penrose pseudo inverse of ϖ , and vice versa. Note that the two-component wavefunction no longer suffers from any constraints such as Eq. (4). Let me explain its physical meaning in the following section.

III. JONES REPRESENTATION AND INNER REFERENCE FRAME

Along with the transformation of the vector wavefunction into the two-component wavefunction via Eq. (13), the position operator $\hat{\mathbf{X}}$ that acts on the vector wavefunction is transformed into [18]

$$\hat{\mathbf{x}} = \varpi^\dagger \hat{\mathbf{X}} \varpi = \hat{\boldsymbol{\xi}} + \hat{\boldsymbol{\Xi}}, \quad (15)$$

which splits into two parts, where $\hat{\boldsymbol{\xi}} = i\nabla$ and

$$\hat{\boldsymbol{\Xi}} = i\varpi^\dagger (\nabla \varpi). \quad (16)$$

The first part $\hat{\boldsymbol{\xi}}$ has the same form as that of $\hat{\mathbf{X}}$ but acts on the two-component wavefunction. Since no constraints such as Eq. (4) exist for the two-component wavefunction, the Cartesian components of $\hat{\boldsymbol{\xi}}$ are independent of one another, obeying the canonical commutation relations,

$$[\hat{\xi}_i, \hat{\xi}_j] = 0. \quad (17)$$

Because the Cartesian components of the momentum $\hat{\mathbf{p}}$ also obey the canonical commutation relations,

$$[\hat{p}_i, \hat{p}_j] = 0, \quad (18)$$

this part is canonically conjugate to the momentum [3], obeying the following canonical commutation relations,

$$[\hat{\xi}_i, \hat{p}_j] = i\hbar\delta_{ij}. \quad (19)$$

The second part $\hat{\boldsymbol{\Xi}}$, also acting on the two-component wavefunction, is Hermitian,

$$\hat{\boldsymbol{\Xi}}^\dagger = -i(\nabla \varpi^\dagger) \varpi = i\varpi^\dagger (\nabla \varpi) = \hat{\boldsymbol{\Xi}},$$

by virtue of property (12). Its Cartesian components commute with one another,

$$[\hat{\Xi}_i, \hat{\Xi}_j] = 0. \quad (20)$$

Moreover, it commutes with the Hamiltonian ω . That is to say, it represents a constant of motion. Taking Eqs. (17) and (20) into account, it is not difficult to find

$$\hat{\mathbf{x}} \times \hat{\mathbf{x}} = i\nabla \times \hat{\boldsymbol{\Xi}}. \quad (21)$$

According to Cohen-Tannoudji *et. al.* [3], $\hat{\mathbf{x}}$ cannot be canonically conjugate to the momentum or, equivalently, the position operator $\hat{\mathbf{X}}$ that acts on the vector wavefunction cannot

be canonically conjugate to the momentum. As a consequence, the OAM given by Eq. (1b) must not obey the standard commutation relations (5b).

The canonical commutation relations (17)-(19) make us have to pay attention to the \mathbf{k} -space gradient operator $\hat{\boldsymbol{\xi}}$. Keep in mind that $\hat{\boldsymbol{\xi}}$ acts on the two-component wavefunction. If regarding $\hat{\boldsymbol{\xi}}$ as representing the position $\boldsymbol{\xi}$ with respect to some reference frame, the two-component wavefunction \tilde{f} is nothing but the \mathbf{k} -space wavefunction in such a reference frame. Its Fourier integral gives the position-space wavefunction

$$\tilde{F}(\boldsymbol{\xi}, t) = \frac{1}{(2\pi)^{3/2}} \int \tilde{f}(\mathbf{k}, t) \exp(i\mathbf{k} \cdot \boldsymbol{\xi}) d^3k \quad (22)$$

in the same reference frame. This is to be compared with the other \mathbf{k} -space gradient operator $\hat{\mathbf{X}}$, the one that represents the position \mathbf{X} with respect to the laboratory reference frame, in which the \mathbf{k} -space wavefunction is the vector wavefunction \mathbf{f} . Even though no position-space wavefunction in the laboratory reference frame can be deduced from the \mathbf{k} -space wavefunction, the position-space electric field in the laboratory reference frame that solves the Maxwell's equations can still be written [1, 3] as $\mathbf{E}(\mathbf{X}, t) + \mathbf{E}^*(\mathbf{X}, t)$, where $\mathbf{E}(\mathbf{X}, t)$ is expressed in terms of the vector wavefunction as

$$\mathbf{E}(\mathbf{X}, t) = \frac{1}{(2\pi)^{3/2}} \int N(\omega) \mathbf{f} \exp(i\mathbf{k} \cdot \mathbf{X}) d^3k \quad (23)$$

and $N(\omega) = \left(\frac{\hbar\omega}{2\varepsilon_0}\right)^{1/2}$. It is important to note that the above mentioned reference frame $\boldsymbol{\xi}$ is not related to the laboratory reference frame simply by a spatial translation. As can be seen from Eq. (15), its origin with respect to the laboratory reference frame is represented by operator $\hat{\boldsymbol{\Xi}}$. Because this operator is dependent on the intrinsic degree of freedom, the helicity, as will be shown in Section V, so identified reference frame will be referred to as the inner reference frame.

For the sake of clarity, the two-component wavefunction that is defined in the inner reference frame will be termed as the Jones wavefunction; the representation constituted by all the Jones wavefunctions as the Jones representation. Accordingly, the vector wavefunction that is defined in the laboratory reference frame will be termed as the Maxwell wavefunction; the representation constituted by all the Maxwell wavefunctions as the Maxwell representation. Substituting Eq. (9) into Eq. (23) and making use of the inverse Fourier transformation of Eq. (22), we find

$$\mathbf{E}(\mathbf{X}, t) = \int \Pi(\mathbf{X} - \boldsymbol{\xi}) \tilde{F}(\boldsymbol{\xi}, t) d^3\xi, \quad (24)$$

where

$$\Pi(\mathbf{X} - \boldsymbol{\xi}) = \frac{1}{(2\pi)^{3/2}} \int N\varpi \exp[i\mathbf{k} \cdot (\mathbf{X} - \boldsymbol{\xi})] d^3k. \quad (25)$$

Eq. (24) is the position-space counterpart of the \mathbf{k} -space relation (9) between the Maxwell wavefunction and the Jones wavefunction. Π is the position-space counterpart of the \mathbf{k} -space quasi unitary matrix ϖ .

IV. CANONICAL QUANTIZATION OF JONES WAVEFUNCTIONS

We have shown that the Jones wavefunction is defined in the inner reference frame and that the position with respect to the inner reference frame is canonically conjugate to the momentum. It is well known that the canonical commutation relations (17)-(19) determine a maximal set of three mutually commuting operators, which describe three canonical variables. If we identify the independent degree of freedom that is described by the two components of the Jones wavefunction, we will readily quantize the Jones wavefunction in terms of the canonical variables. In the following I will show that it is the helicity rather than the spin that appears as the independent intrinsic degree of freedom.

A. Helicity is independent of the canonical variables

The spin operator (1a) in the Maxwell representation is transformed by the quasi unitary matrix ϖ into

$$\hat{\mathbf{s}} = \varpi^\dagger \hat{\mathbf{S}} \varpi = \hbar \varpi^\dagger \hat{\boldsymbol{\Sigma}} \varpi \quad (26)$$

in the Jones representation. We decompose the vector operator $\hat{\boldsymbol{\Sigma}}$ in the local coordinate system uvw as

$$\hat{\boldsymbol{\Sigma}} = \mathbf{u}(\mathbf{u}^T \hat{\boldsymbol{\Sigma}}) + \mathbf{v}(\mathbf{v}^T \hat{\boldsymbol{\Sigma}}) + \mathbf{w}(\mathbf{w}^T \hat{\boldsymbol{\Sigma}}).$$

Substituting it into Eq. (26) and making use of Eqs. (8), we get

$$\hat{\mathbf{s}} = \mathbf{w} \hbar \hat{\sigma}_3, \quad (27)$$

where

$$\hat{\sigma}_3 = \varpi^\dagger (\mathbf{w}^T \hat{\boldsymbol{\Sigma}}) \varpi = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (28)$$

is one of the Pauli matrices. Eq. (27) reveals an important characteristic that the spin of the photon lies entirely along the direction of the wavevector [19]. It is important to note that due to the \mathbf{w} -dependence, the spin is not independent of the canonical variables. As a matter of fact, it is the helicity that is the independent intrinsic degree of freedom. This is because the helicity is represented by the constant Pauli matrix $\hat{\sigma}_3$. It has eigenvalues $\sigma = \pm 1$, corresponding to eigenfunctions

$$\tilde{\alpha}_{+1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \tilde{\alpha}_{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}, \quad (29)$$

respectively.

It deserves pointing out that the helicity operator $\mathbf{w}^T \hat{\Sigma}$ in the Maxwell representation depends on the wavevector. Only in the Jones representation can the helicity operator be expressed independently of the canonical variables. This is an interesting phenomenon. It probably implies that the notion of the inner reference frame is required by the separation of the helicity from the canonical variables. After all, the Jones wavefunction is defined in the inner reference frame.

B. Complete set of mode functions in the Jones representation

It is now clear that we have a maximal set of four commuting operators to quantize the Jones wavefunction. One is the Pauli matrix (28) describing the helicity. The other three describe three commuting canonical variables. By this we mean that we have a set of four quantum numbers to characterize a complete set of mode functions in the Jones representation. Depending on the different choices of three commuting canonical variables, we can have different sets of four quantum numbers and therefore different complete sets of mode functions. Here I put forward three kinds that are commonly used.

1. Plane-wave modes

Since the momentum commutes with the Hamiltonian ω , the canonical commutation relations (18) allow us to choose the three Cartesian components of the momentum as the commuting canonical variables. Denoting by $\hbar \mathbf{k}_0$ the eigen momentum, we have for the

complete set of mode functions,

$$\tilde{f}_\tau(\mathbf{k}) = \tilde{\alpha}_\sigma \delta^3(\mathbf{k} - \mathbf{k}_0), \quad (30)$$

where $\tilde{\alpha}_\sigma$ is given by Eq. (29) and τ is a collective index to stand for the four quantum numbers: $\tau = (\sigma, \mathbf{k}_0)$. The orthonormality relation reads

$$\int \tilde{f}_{\tau'}^\dagger \tilde{f}_\tau d^3k = \delta_{\sigma'\sigma} \delta^3(\mathbf{k}'_0 - \mathbf{k}_0), \quad (31)$$

where $\tau' = (\sigma', \mathbf{k}'_0)$.

2. Spherical surface harmonics

The canonical commutation relations (17)-(19) defines the following canonical OAM,

$$\hat{\boldsymbol{\lambda}} = -\hbar \mathbf{k} \times \hat{\boldsymbol{\xi}}, \quad (32)$$

which obeys the standard commutation relations of the angular momentum,

$$[\hat{\lambda}_i, \hat{\lambda}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{\lambda}_k. \quad (33)$$

It represents the OAM about the origin of the inner reference frame. Noticing that $\hat{\boldsymbol{\lambda}}$ commutes with the Hamiltonian ω ,

$$[\hat{\boldsymbol{\lambda}}, \omega] = 0, \quad (34)$$

the second choice of three commuting canonical variables can be ω , $\hat{\boldsymbol{\lambda}}^2$, and $\hat{\lambda}_3 = -i\hbar \frac{\partial}{\partial \varphi}$. It is well known that the simultaneous normalized eigenfunctions of $\hat{\boldsymbol{\lambda}}^2$ and $\hat{\lambda}_3$ in \mathbf{k} -space are the spherical surface harmonics [3],

$$Y_{\lambda m}(\mathbf{w}) = \left\{ \frac{2\lambda + 1}{4\pi} \frac{(\lambda - m)!}{(\lambda + m)!} \right\}^{1/2} P_\lambda^m(\cos \vartheta) e^{im\varphi},$$

which satisfy the following eigenvalue equations,

$$\hat{\boldsymbol{\lambda}}^2 Y_{\lambda m} = \lambda(\lambda + 1) \hbar^2 Y_{\lambda m}, \quad \lambda = 0, 1, 2, \dots \quad (35a)$$

$$\hat{\lambda}_3 Y_{\lambda m} = m \hbar Y_{\lambda m}, \quad m = 0, \pm 1, \pm 2, \dots \pm \lambda. \quad (35b)$$

Their orthonormality relation assumes the form

$$\int Y_{\lambda' m'}^* Y_{\lambda m} \sin \vartheta d\vartheta d\varphi = \delta_{\lambda' \lambda} \delta_{m' m}.$$

As a result, the expected mode functions in this case take the form of

$$\tilde{f}_\tau(\mathbf{k}) = \tilde{\alpha}_\sigma \frac{\delta(k - k_0)}{\sqrt{k_0 \omega_0}} Y_{\lambda m}(\mathbf{w}), \quad (36)$$

where $\tau = (\sigma, \omega_0, \lambda, m)$ and $k_0 = \omega_0/c$. They form a complete set and have the following orthonormality relation,

$$\int \tilde{f}_{\tau'}^\dagger \tilde{f}_\tau d^3k = \delta_{\sigma'\sigma} \delta(\omega'_0 - \omega_0) \delta_{\lambda'\lambda} \delta_{m'm}, \quad (37)$$

where $\tau' = (\sigma', \omega'_0, \lambda', m')$.

3. Diffraction-free modes

Noticing that \hat{p}_3 and $\hat{\lambda}_3$ commute, the third choice of three commuting canonical variables can be ω , \hat{p}_3 , and $\hat{\lambda}_3$. \hat{p}_3 and $\hat{\lambda}_3$ have the following simultaneous normalized eigenfunctions in circular cylindrical coordinates,

$$X_{k_{z0}m}(\varphi, k_z) = \frac{1}{\sqrt{2\pi}} \delta(k_z - k_{z0}) e^{im\varphi}, \quad m = 0, \pm 1, \pm 2 \dots$$

with eigenvalues $k_{z0}\hbar$ and $m\hbar$, respectively. The corresponding orthonormal mode functions that span the Jones representation are given by

$$\tilde{f}_\tau(\mathbf{k}) = \tilde{\alpha}_\sigma \frac{\sqrt{\omega_0}}{ck_{\rho 0}} \delta(k_\rho - k_{\rho 0}) X_{k_{z0}m}(\varphi, k_z), \quad (38)$$

which describe diffraction-free modes at position space [20], where $\tau = (\sigma, \omega_0, k_{z0}, m)$ and $k_{\rho 0} = (k_0^2 - k_{z0}^2)^{1/2}$. They satisfy the following orthonormality relation,

$$\int \tilde{f}_{\tau'}^\dagger \tilde{f}_\tau d^3k = \delta_{\sigma'\sigma} \delta(\omega'_0 - \omega_0) \delta(k'_{z0} - k_{z0}) \delta_{m'm}, \quad (39)$$

where $\tau' = (\sigma', \omega'_0, k'_{z0}, m')$.

V. DEGREE OF FREEDOM TO SPECIFY THE INNER REFERENCE FRAME

From the transversality condition we introduced the Jones wavefunction and successfully quantized it in a canonical way. The problem that we have to face up with is that the transversality condition alone is not able to determine the inner reference frame in which the Jones wavefunction is defined. The reason is as follows. The operator (16) for the origin

of the inner reference frame with respect to the laboratory reference frame is expressed in terms of the quasi unitary matrix ϖ , but Eqs. (8) that guarantee the transversality condition cannot fully determine ϖ up to a rotation about the wavevector [19]. In this section, I will identify a new degree of freedom to complement the determination of the inner reference frame and will explore its physical significance from two different aspects.

A. A new degree of freedom

It was once shown [21] that the two orthonormal vectors \mathbf{u} and \mathbf{v} that make up the quasi unitary matrix (10) can be fully determined by an independent unit vector in addition to the wavevector. Indeed, denoting by \mathbf{I} a constant unit vector, it is not difficult to show that the following two unit vectors do satisfy Eqs. (8),

$$\mathbf{u}_{\mathbf{I}} = \mathbf{v}_{\mathbf{I}} \times \frac{\mathbf{k}}{k}, \quad \mathbf{v}_{\mathbf{I}} = \frac{\mathbf{I} \times \mathbf{k}}{|\mathbf{I} \times \mathbf{k}|}. \quad (40)$$

Considering this constant unit vector, the quasi unitary matrix (10) should be rewritten as

$$\varpi_{\mathbf{I}} = (\mathbf{u}_{\mathbf{I}} \quad \mathbf{v}_{\mathbf{I}}). \quad (41)$$

Correspondingly, Eqs. (12) and (14) for the quasi unitarity become

$$\varpi_{\mathbf{I}}^\dagger \varpi_{\mathbf{I}} = I_2, \quad (42a)$$

$$\varpi_{\mathbf{I}} \varpi_{\mathbf{I}}^\dagger = I_3, \quad (42b)$$

respectively.

Substituting Eq. (41) into Eq. (16) and denoting the resultant by $\hat{\Xi}_{\mathbf{I}}$, we get for the operator of the origin of the inner reference frame,

$$\hat{\Xi}_{\mathbf{I}} = \hat{\sigma}_3 \frac{\mathbf{I} \cdot \mathbf{k}}{k|\mathbf{I} \times \mathbf{k}|} \mathbf{v}_{\mathbf{I}}, \quad (43)$$

where $\hat{\sigma}_3$ is the helicity operator (28). It shows that the unit vector \mathbf{I} does appear as an independent degree of freedom to determine the inner reference frame. Specifically, the same as $\mathbf{v}_{\mathbf{I}}$, $\hat{\Xi}_{\mathbf{I}}$ is perpendicular to \mathbf{I} . Due to this, we rewrite Eq. (15) as

$$\hat{\mathbf{x}} = \hat{\xi} + \hat{\Xi}_{\mathbf{I}}. \quad (44)$$

Another important characteristic that Eq. (43) reveals is that the origin of the inner reference frame depends on the helicity.

B. Analogy to the classical gauge degree of freedom

Taking the degree of freedom \mathbf{I} into account, we rewrite Eq. (13) as

$$\tilde{f}_{\mathbf{I}} = \varpi_{\mathbf{I}}^{\dagger} \mathbf{f}. \quad (45)$$

It means that the Jones wavefunction for a particular radiation field is not uniquely determined, depending on the choice of the inner reference frame that is specified by the degree of freedom \mathbf{I} . This situation is analogous to the gauge degree of freedom of the electromagnetic potentials in classical theory [22]. To show this, let us examine how the choice of the unit vector \mathbf{I} affects the Jones wavefunction.

Consider a different constant unit vector, say \mathbf{I}' . The corresponding Jones wavefunction for the same radiation field is given by

$$\tilde{f}_{\mathbf{I}'} = \varpi_{\mathbf{I}'}^{\dagger} \mathbf{f}, \quad (46)$$

where $\varpi_{\mathbf{I}'} = (\mathbf{u}_{\mathbf{I}'} \ \mathbf{v}_{\mathbf{I}'})$ and

$$\mathbf{u}_{\mathbf{I}'} = \mathbf{v}_{\mathbf{I}'} \times \frac{\mathbf{k}}{k}, \quad \mathbf{v}_{\mathbf{I}'} = \frac{\mathbf{I}' \times \mathbf{k}}{|\mathbf{I}' \times \mathbf{k}|}.$$

As remarked earlier, the orthonormal vectors $\mathbf{u}_{\mathbf{I}'}$ and $\mathbf{v}_{\mathbf{I}'}$ that make up the new transformation matrix $\varpi_{\mathbf{I}'}$ are related to the old orthonormal vectors $\mathbf{u}_{\mathbf{I}}$ and $\mathbf{v}_{\mathbf{I}}$ by a rotation about \mathbf{w} , which can be expressed as follows,

$$\mathbf{u}_{\mathbf{I}'} = \mathbf{u}_{\mathbf{I}} \cos \phi + \mathbf{v}_{\mathbf{I}} \sin \phi, \quad (47a)$$

$$\mathbf{v}_{\mathbf{I}'} = -\mathbf{u}_{\mathbf{I}} \sin \phi + \mathbf{v}_{\mathbf{I}} \cos \phi, \quad (47b)$$

where ϕ denotes the pertinent rotation angle. These two equations can be rewritten in terms of the transformation matrices $\varpi_{\mathbf{I}'}$ and $\varpi_{\mathbf{I}}$ as

$$\varpi_{\mathbf{I}'} = \varpi_{\mathbf{I}} D, \quad (48)$$

where $D = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}$ is the rotation matrix, which can be expressed in terms of the helicity operator $\hat{\sigma}_3$ as

$$D = \exp(-i\hat{\sigma}_3\phi). \quad (49)$$

Substituting Eq. (48) into Eq. (46) and making use of Eqs. (45) and (49), we have

$$\tilde{f}_{\mathbf{I}'} = D^\dagger \tilde{f}_{\mathbf{I}} = \exp(i\hat{\sigma}_3\phi)\tilde{f}_{\mathbf{I}}. \quad (50)$$

The rotation angle ϕ is determined as follows.

The barycenter of the radiation field with respect to the laboratory reference frame should not depend on the choice of \mathbf{I} . This means that

$$\tilde{f}_{\mathbf{I}'}^\dagger(\hat{\Xi}_{\mathbf{I}'} + i\nabla)\tilde{f}_{\mathbf{I}'} = \tilde{f}_{\mathbf{I}}^\dagger(\hat{\Xi}_{\mathbf{I}} + i\nabla)\tilde{f}_{\mathbf{I}}, \quad (51)$$

by virtue of Eq. (44), where

$$\hat{\Xi}_{\mathbf{I}'} = i\varpi_{\mathbf{I}'}^\dagger(\nabla\varpi_{\mathbf{I}'}) = \hat{\sigma}_3 \frac{\mathbf{I}' \cdot \mathbf{k}}{k|\mathbf{I}' \times \mathbf{k}|} \mathbf{v}_{\mathbf{I}'}.$$

Upon substituting Eq. (50) into Eq. (51), we find

$$\tilde{f}_{\mathbf{I}}^\dagger(\hat{\Xi}_{\mathbf{I}'} - \hat{\Xi}_{\mathbf{I}})\tilde{f}_{\mathbf{I}} = \tilde{f}_{\mathbf{I}}^\dagger(\nabla\phi)\hat{\sigma}_3\tilde{f}_{\mathbf{I}},$$

which is equivalent to

$$\hat{\Xi}_{\mathbf{I}'} - \hat{\Xi}_{\mathbf{I}} = (\nabla\phi)\hat{\sigma}_3, \quad (52)$$

due to the arbitrariness of the Maxwell wavefunction \mathbf{f} and of the unit vector \mathbf{I} . Eq. (52) determines the rotation angle ϕ under the change of the unit vector \mathbf{I} .

It is well known in classical electromagnetic theory [22] that any particular radiation field in free space can be expressed in terms of four gauge potentials. In that expression, we have a gauge degree of freedom in the sense that the radiation field is invariant under a gauge transformation. Because only two of the four gauge potentials are truly independent as is extensively discussed in Ref. [3], the Jones wavefunction given by Eq. (45) can be regarded as the quantum analog to the gauge potentials; the unit vector \mathbf{I} as the quantum analog to the gauge degree of freedom; Eq. (50) as the quantum analog to the gauge transformation, where the “gauge function” ϕ is determined by Eq. (52). In fact, according to Eq. (42b), we may change Eq. (45) into

$$\mathbf{f} = \varpi_{\mathbf{I}}\tilde{f}_{\mathbf{I}}. \quad (53)$$

It states that the Maxwell wavefunction \mathbf{f} to describe the radiation field in the laboratory reference frame can always be expressed in terms of a Jones wavefunction that is specified by the degree of freedom \mathbf{I} .

C. Barycenter theory for eigen excitations

Now we are ready to explain the physical significance of the degree of freedom \mathbf{I} from the quantum point of view. To this end, we expand the Jones wavefunction (45) in a complete orthonormal set of mode functions as

$$\tilde{f}_{\mathbf{I}} = \sum_{\tau} a_{\mathbf{I},\tau} \tilde{f}_{\tau}, \quad (54)$$

where it is assumed that the summation over τ includes the integration over continuous quantum numbers. The \mathbf{I} -dependence of the expansion coefficient implies that apart from the quantum numbers τ , the unit vector \mathbf{I} also plays role in characterizing the eigen excitation of the radiation field.

To see what is meant by this, we substitute Eq. (54) into Eq. (53) to get

$$\mathbf{f} = \sum_{\tau} a_{\mathbf{I},\tau} \mathbf{f}_{\mathbf{I},\tau}, \quad (55)$$

where

$$\mathbf{f}_{\mathbf{I},\tau} = \varpi_{\mathbf{I}} \tilde{f}_{\tau}. \quad (56)$$

Eq. (55) predicts that all the Maxwell wavefunctions $\mathbf{f}_{\mathbf{I},\tau}$ that are given by Eq. (56) with a specific unit vector \mathbf{I} form a complete orthonormal set, which spans the Maxwell representation. This is indeed the case as one may easily see from the quasi unitarity (42a) as well as the orthonormality relation (31), (37), or (39). Eq. (56) tells how one can obtain a complete set of mode functions in the Maxwell representation from any complete set of mode functions in the Jones representation. Eq. (55) is thus interpreted as expanding the Maxwell wavefunction of a radiation field in such a complete set. Each mode function $\mathbf{f}_{\mathbf{I},\tau}$ in the set describes an eigen excitation of the radiation field, showing that the degree of freedom \mathbf{I} does play its role in characterizing the eigen excitation. This conclusion can be understood as follows.

The canonical position that is conjugate to the canonical momentum is measured with respect to the inner reference frame. Besides, only in the inner reference frame can the helicity be regarded as the intrinsic degree of freedom that is independent of the canonical variables. Therefore, the quantum numbers τ can only determine the properties of the eigen excitation with respect to its own inner reference frame. However, a physically meaningful eigen excitation requires that its inner reference frame be unambiguously specified with

respect to the laboratory reference frame. This is done by the complementary degree of freedom, the unit vector \mathbf{I} . In fact, for an eigen excitation that is associated with one previously mentioned mode function \tilde{f}_τ via Eq. (56), the degree of freedom \mathbf{I} specifies its barycenter with respect to the laboratory reference frame via Eq. (43) as we show below.

In any previously mentioned mode function \tilde{f}_τ , the expectation value of $\hat{\boldsymbol{\xi}}$ vanishes,

$$\langle \hat{\boldsymbol{\xi}} \rangle_\tau \equiv \frac{\int \tilde{f}_\tau^\dagger \hat{\boldsymbol{\xi}} \tilde{f}_\tau d^3k}{\int \tilde{f}_\tau^\dagger \tilde{f}_\tau d^3k} = 0, \quad (57)$$

as one may easily check. Consequently, the inner reference frame for such an eigen excitation reduces to its barycenter reference frame in the sense that its barycenter is located exactly at the origin of the inner reference frame. In other words, the operator $\hat{\boldsymbol{\Xi}}_{\mathbf{I}}$ that is given by Eq. (43) represents the barycenter of the eigen excitation $\mathbf{f}_{\mathbf{I},\tau}$ with respect to the laboratory reference frame. Given the quantum numbers τ , the degree of freedom \mathbf{I} will specify the barycenter of the associated eigen excitation. A discussion of the effect of \mathbf{I} on the barycenter of diffraction-free modes can be found in Ref. [23]. Let us examine how the choice of the unit vector \mathbf{I} affects the eigen excitation.

As previously mentioned, the mode function \tilde{f}_τ in the Jones representation is the eigenfunction of the helicity operator $\hat{\sigma}_3$, having eigenvalue σ ,

$$\hat{\sigma}_3 \tilde{f}_\tau = \sigma \tilde{f}_\tau. \quad (58)$$

For a specific constant unit vector \mathbf{I} , the mode function $\mathbf{f}_{\mathbf{I},\tau}$ in the Maxwell representation is given by Eq. (56). For a different constant unit vector, say \mathbf{I}' , the corresponding mode function in the Maxwell representation is given by

$$\mathbf{f}_{\mathbf{I}',\tau} = \varpi_{\mathbf{I}'} \tilde{f}_\tau. \quad (59)$$

Substituting Eqs. (48) and (49) into Eq. (59) and noticing the eigenvalue equation (58), we get

$$\mathbf{f}_{\mathbf{I}',\tau} = \exp(-i\sigma\phi) \mathbf{f}_{\mathbf{I},\tau}. \quad (60)$$

It shows that when the unit vector \mathbf{I} is changed with the quantum numbers τ remaining unchanged, the mode function in the Maxwell representation acquires a phase factor. Depending on the helicity σ as well as on the wavevector through the rotation angle ϕ , this phase factor will affect the vector nature of the position-space electric field in the laboratory reference frame as can be seen from Eq. (23). A discussion of the effect of \mathbf{I} on the vector nature of diffraction-free modes can be found in Ref. [20].

VI. QUANTUM THEORY FOR THE ANGULAR MOMENTUM

Having introduced the Jones representation in which the wavefunction is not constrained by any conditions, we are in a position to investigate the properties of photon's angular momenta by means of straightforward calculations in the Jones representation.

A. Spin

The spin operator in the Jones representation is given by Eq. (27). Since \mathbf{w} is a c-number vector in the \mathbf{k} -space, different Cartesian components of the spin commute with one another,

$$[\hat{s}_i, \hat{s}_j] = 0. \quad (61)$$

This is in agreement with the observation of van Enk and Nienhuis [4]. Besides, the spin is a constant of motion, because the momentum and the helicity are constants of motion.

As a quantization condition, a commutation relation should not depend on the choice of the representation. So it is expected from Eq. (61) that the spin operator $\hat{\mathbf{S}}$ in the Maxwell representation obeys

$$[\hat{S}_i, \hat{S}_j] = 0. \quad (62)$$

Indeed, considering the quasi unitarities (42), it is not difficult to find from Eqs. (26) and (27) that $\hat{\mathbf{S}} = \hbar \mathbf{w}(\varpi_{\mathbf{I}} \hat{\sigma}_3 \varpi_{\mathbf{I}}^\dagger)$. Since

$$\varpi_{\mathbf{I}} \hat{\sigma}_3 \varpi_{\mathbf{I}}^\dagger = \mathbf{w}^T \hat{\Sigma}$$

as can be seen from Eq. (28), we finally have

$$\hat{\mathbf{S}} = \hbar \mathbf{w}(\mathbf{w}^T \hat{\Sigma}).$$

It does obey the commutation relations (62).

B. Orbital angular momentum

The same as the operator (44) of the position with respect to the laboratory reference frame, the operator of the OAM about the origin of the laboratory reference frame also splits into two parts,

$$\hat{\mathbf{I}} = \varpi_{\mathbf{I}}^\dagger \hat{\mathbf{L}} \varpi_{\mathbf{I}} = \hat{\boldsymbol{\lambda}} + \hat{\boldsymbol{\Lambda}}, \quad (63)$$

where $\hat{\boldsymbol{\Lambda}}$ is given by Eq. (32) and

$$\hat{\boldsymbol{\Lambda}} \equiv \hbar \boldsymbol{\Xi}_{\mathbf{I}} \times \mathbf{k} = \hbar \hat{\sigma}_3 \frac{\mathbf{I} \cdot \mathbf{k}}{|\mathbf{I} \times \mathbf{k}|} \mathbf{u}_{\mathbf{I}}. \quad (64)$$

The first part $\hat{\boldsymbol{\Lambda}}$ is the OAM about the origin of the inner reference frame. The second part $\hat{\mathbf{A}}$ is the OAM of the origin of the inner reference frame about the origin of the laboratory reference frame. Obviously, it is dependent on the helicity. This not only explains why the entire OAM about the origin of the laboratory reference frame depends on the helicity [14], but also helps us to understand why the total angular momentum cannot be generally separated into helicity-dependent spin and helicity-independent OAM [12]. Like $\hat{\boldsymbol{\Xi}}_{\mathbf{I}}$, $\hat{\mathbf{A}}$ commutes with the Hamiltonian,

$$[\hat{\mathbf{A}}, \omega] = 0, \quad (65)$$

and its Cartesian components commute with one another,

$$[\hat{\Lambda}_i, \hat{\Lambda}_j] = 0. \quad (66)$$

From Eqs. (34) and (65) it follows that the entire OAM is a constant of motion,

$$[\hat{\mathbf{I}}, \omega] = 0.$$

Noticing Eqs. (33) and (66), straightforward but time-consuming calculations yield

$$[\hat{l}_i, \hat{l}_j] = i\hbar \sum_k \epsilon_{ijk} (\hat{l}_k - \hat{s}_k). \quad (67)$$

They are exactly the commutation relations (6b) that were found by van Enk and Nienhuis [4]. But we arrived at them without resorting to the second quantization.

It is noted that the first part, the canonical OAM, does not commute with the spin,

$$[\hat{\lambda}_i, \hat{s}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{s}_k,$$

though the second part does. In other words, the spin is not at all a degree of freedom that is independent of the canonical OAM. As a result, the spin does not commute with the entire OAM,

$$[\hat{l}_i, \hat{s}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{s}_k. \quad (68)$$

Eqs. (44) and (63) demonstrate that the origin of the inner reference frame can be regarded as the generalization of the barycenter of the classical mechanics [24] in the following

sense. The position vector with respect to the laboratory reference frame is the position vector with respect to the inner reference frame plus the position vector of the origin of the inner reference frame with respect to the laboratory reference frame; The OAM about the origin of the laboratory reference frame is the OAM about the origin of the inner reference frame plus the OAM of the origin of the inner reference frame about the origin of the laboratory reference frame. In the case of eigen excitations, the origin of the inner reference frame appears simply as the barycenter.

C. Total angular momentum

From Eqs. (27), (63), and (64), it is easy to obtain for the operator of the total angular momentum,

$$\hat{\mathbf{j}} = \hat{\mathbf{s}} + \hat{\mathbf{l}} = \hat{\boldsymbol{\lambda}} + \hbar \hat{\sigma}_3 \frac{\mathbf{I} \times \mathbf{v}_\mathbf{I}}{\mathbf{I} \cdot \mathbf{u}_\mathbf{I}}, \quad (69)$$

which shows a very interesting property that the component of $\hat{\mathbf{j}}$ in the direction of \mathbf{I} is equal to the component of $\hat{\boldsymbol{\lambda}}$ in the same direction. From Eqs. (61), (67), and (68) it follows that the total angular momentum obeys the standard commutation relations,

$$[\hat{j}_i, \hat{j}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{j}_k, \quad (70)$$

as it should do.

VII. CONCLUSIONS AND REMARKS

In summary, a quantum theory for the angular momentum of the photon is formulated by introducing the so-called Jones representation. The wavefunction in the Jones representation, the Jones wavefunction, is defined in the inner reference frame. This is compared with the Maxwell wavefunction that is defined in the laboratory reference frame. The position vector with respect to the laboratory reference frame is the position vector $\hat{\boldsymbol{\xi}}$ with respect to the inner reference frame plus the position vector $\hat{\boldsymbol{\Xi}}_\mathbf{I}$ of the origin of the inner reference frame with respect to the laboratory reference frame. The OAM about the origin of the laboratory reference frame is the OAM about the origin of the inner reference frame plus the OAM of the origin of the inner reference frame about the origin of the laboratory reference frame. So the origin of the inner reference frame can be regarded as the generalization of

the concept of classical barycenter. Because the position with respect to the inner reference frame is canonically conjugate to the momentum, the Jones wavefunction can be canonically quantized. In addition, a new degree of freedom that appears as a unit vector \mathbf{I} is identified to specify the inner reference frame. To one complete set of mode functions in the Jones representation that is characterized by a set of four quantum numbers τ , there correspond an infinite number of complete sets of mode functions in the Maxwell representation. Each of them is distinguished from others by a specific value of the degree of freedom \mathbf{I} . For a radiation field that is described by a particular Maxwell wavefunction, the new degree of freedom is analogous to the classical gauge degree of freedom and the Jones wavefunction is analogous to the classical gauge potentials. On the other hand, for a complete set of eigen excitations corresponding to a given complete set of mode functions in the Maxwell representation, the new degree of freedom plays the role of specifying the barycenter of each eigen excitation with respect to the laboratory reference frame.

Generally speaking, if the Jones wavefunction \tilde{f} of a photon state gives rise to vanishing expectation value of $\hat{\xi}$, the inner reference frame reduces to the barycenter reference frame of that state; the operator $\hat{\Xi}_{\mathbf{I}}$ given by Eq. (43) represents the barycenter with respect to the laboratory reference frame. It is noted that the barycenter is transverse because the vector operator $\hat{\Xi}_{\mathbf{I}}$ is perpendicular to the wavevector. If an optical process deflects only the unit vector \mathbf{I} without changing the Jones wavefunction \tilde{f} in the barycenter reference frame, this process will only shift the transverse barycenter of the photon state. The so-called spin Hall effect [25] is just such a process [26].

At last, it is worth pointing out that when the second quantization is applied to the Maxwell or Jones wavefunction, particular attention should be paid to the \mathbf{I} -dependence of the annihilation operator that corresponds to the expansion coefficient in Eq. (54) or (55).

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